

# One more Turán number and Ramsey number for the loose 3-uniform path of length three

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## Abstract

Let  $P$  denote a 3-uniform hypergraph consisting of 7 vertices  $a, b, c, d, e, f, g$  and 3 edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, g\}$ . It is known that the  $r$ -color Ramsey number for  $P$  is  $R(P; r) = r + 6$  for  $r \leq 9$ . The proof of this result relies on a careful analysis of the Turán numbers for  $P$ . In this paper, we refine this analysis further and compute the fifth order Turán number for  $P$ , for all  $n$ . Using this number for  $n = 16$ , we confirm the formula  $R(P; 10) = 16$ .

## 1 Introduction

For the sake of brevity, 3-uniform hypergraphs will be called here *3-graphs*. Given a family of 3-graphs  $\mathcal{F}$ , we say that a 3-graph  $H$  is  $\mathcal{F}$ -free if for all  $F \in \mathcal{F}$  we have  $H \not\supseteq F$ .

For a family of 3-graphs  $\mathcal{F}$  and an integer  $n \geq 1$ , the *Turán number of the 1st order*, that is, the ordinary Turán number, is defined as

$$\text{ex}(n; \mathcal{F}) = \text{ex}^{(1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every  $n$ -vertex  $\mathcal{F}$ -free 3-graph with  $\text{ex}^{(1)}(n; \mathcal{F})$  edges is called *1-extremal for  $\mathcal{F}$* . We denote by  $\text{Ex}^{(1)}(n; \mathcal{F})$  the family of all, pairwise non-isomorphic,  $n$ -vertex 3-graphs which are 1-extremal for  $\mathcal{F}$ . Further, for an integer  $s \geq 1$ , the *Turán number of the  $(s+1)$ -st order* is defined as

$$\begin{aligned} \text{ex}^{(s+1)}(n; \mathcal{F}) &= \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and} \\ &\quad \forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\supseteq H'\}, \end{aligned}$$

if such a 3-graph  $H$  exists. Note that if  $\text{ex}^{(s+1)}(n; \mathcal{F})$  exists then, by definition,

$$\text{ex}^{(s+1)}(n; \mathcal{F}) < \text{ex}^{(s)}(n; \mathcal{F}). \quad (1)$$

An  $n$ -vertex  $\mathcal{F}$ -free 3-graph  $H$  is called  $(s+1)$ -*extremal for  $\mathcal{F}$*  if  $|E(H)| = \text{ex}^{(s+1)}(n; \mathcal{F})$  and  $\forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'$ ; we denote by  $\text{Ex}^{(s+1)}(n; \mathcal{F})$  the family of  $n$ -vertex 3-graphs which are  $(s+1)$ -extremal for  $\mathcal{F}$ . In the case when  $\mathcal{F} = \{F\}$ , we will write  $F$  instead of  $\{F\}$ .

A *loose 3-uniform path of length 3* is a 3-graph  $P$  consisting of 7 vertices, say,  $a, b, c, d, e, f, g$ , and 3 edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, g\}$ . The *Ramsey number*  $R(P; r)$  is the least integer  $n$  such that every  $r$ -coloring of the edges of the complete 3-graph  $K_n$  results in a monochromatic copy of  $P$ . Gyárfas and Raeisi [6] proved, among many other results, that  $R(P; 2) = 8$ . (This result was later extended to loose paths of arbitrary lengths, but still  $r = 2$ , in [13].) Then Jackowska [9] showed that  $R(P; 3) = 9$  and  $r + 6 \leq R(P; r)$  for all  $r \geq 3$ . In turn, in [10], [11], and [15], the Turán numbers of the first four orders,  $\text{ex}^{(i)}(n; P)$ ,  $i = 1, 2, 3, 4$ , have been determined for all feasible values of  $n$ . Using these numbers, in [11] and [15], we were able to compute the Ramsey numbers  $R(P; r)$  for  $4 \leq r \leq 9$ .

**Theorem 1** ([6, 9, 11, 15]). *For all  $r \leq 9$ ,  $R(P; r) = r + 6$ .*

In this paper we determine, for all  $n \geq 7$ , the Turán numbers for  $P$  of the fifth order,  $\text{ex}^{(5)}(n; P)$ . This allows us to compute one more Ramsey number.

**Theorem 2.**  $R(P; 10) = 16$ .

It seems that in order to make a further progress in computing the Ramsey numbers  $R(P; r)$ ,  $r \geq 11$ , one would need to determine still higher order Turán numbers  $\text{ex}^{(s)}(n; P)$ , at least for some small values of  $n$ .

Throughout, we denote by  $S_n$  the 3-graph on  $n$  vertices and with  $\binom{n-1}{2}$  edges, in which one vertex, referred to as *the center*, forms edges with all pairs of the remaining vertices. Every sub-3-graph of  $S_n$  without isolated vertices is called *a star*, while  $S_n$  itself is called *the full star*. We denote by  $C$  *the triangle*, that is, a 3-graph with six vertices  $a, b, c, d, e, f$  and three edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, a\}$ . Finally,  $M$  stands for a pair of disjoint edges. For a given 3-graph  $H$  and a vertex  $v \in V(G)$  we denote by  $\deg_H(v)$  the number of edges in  $H$  containing  $v$ .

In the next section we state some known and new results on Turán numbers for  $P$ , including Theorem 11 which provides a complete formula for  $\text{ex}^{(5)}(n; P)$ . We also define conditional Turán numbers and quote from [11] and [14] some useful lemmas about the conditional Turán numbers with respect to  $P$ ,  $C$ ,  $M$ . Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 11.

## 2 Turán numbers

We restrict ourselves exclusively to the case  $k = 3$  only. A celebrated result of Erdős, Ko, and Rado [2] asserts, in the case of  $k = 3$ , that for  $n \geq 6$ ,  $\text{ex}^{(1)}(n; M) = \binom{n-1}{2}$ . Moreover, for  $n \geq 7$ ,  $\text{Ex}^{(1)}(n; M) = \{S_n\}$ . We will need the higher order versions of this Turán number, together with its extremal families. The second of these numbers has been found by Hilton and Milner, [8] (see [4] and [14] for a simple proof). For a given

set of vertices  $V$ , with  $|V| = n \geq 7$ , let us define two special 3-graphs. Let  $x, y, z, v \in V$  be four different vertices of  $V$ . We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset \right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2 \right\}.$$

Note that for  $i \in \{1, 2\}$ ,  $M \not\subset G_i(n)$  and  $|G_i(n)| = 3n - 8$ .

**Theorem 3** ([8]). *For  $n \geq 7$ ,  $\text{ex}^{(2)}(n; M) = 3n - 8$  and  $\text{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$ .*

Later, we will use the fact that  $C \subset G_i(n) \not\supset P$ ,  $i = 1, 2$ .

Recently, the third order Turán number for  $M$  has been established for general  $k$  by Han and Kohayakawa in [7]. Let  $G_3(n)$  be the 3-graph on  $n$  vertices, with distinguished vertices  $x, y_1, y_2, z_1, z_2$  whose edge set consists of all edges spanned by  $x, y_1, y_2, z_1, z_2$  except for  $\{y_1, y_2, z_i\}$ ,  $i = 1, 2$ , and all edges of the form  $\{x, z_i, v\}$ ,  $i = 1, 2$ , where  $v \notin \{x, y_1, y_2, z_1, z_2\}$ .

**Theorem 4** ([7]). *For  $n \geq 7$ ,  $\text{ex}^{(3)}(n; M) = 2n - 2$  and  $\text{Ex}^{(3)}(n; M) = \{G_3(n)\}$ .*

For  $k = 3$  we were able to take the next step and determine the next Turán number for  $M$ .

**Theorem 5** ([14]). *For  $n \geq 7$ ,  $\text{ex}^{(4)}(n; M) = n + 4$ .*

The number  $\binom{n-1}{2}$  serves as the Turán number for two other 3-graphs,  $C$  and  $P$ . The Turán number  $\text{ex}^{(1)}(n; C)$  has been determined in [3] for  $n \geq 75$  and later for all  $n$  in [1].

**Theorem 6** ([1]). *For  $n \geq 6$ ,  $\text{ex}^{(1)}(n; C) = \binom{n-1}{2}$ . Moreover, for  $n \geq 8$ ,  $\text{Ex}^{(1)}(n; C) = \{S_n\}$ .*

In [10], we filled an omission of [5] and [12] and calculated  $\text{ex}^{(1)}(n; P)$  for all  $n$ . Given two 3-graphs  $F_1$  and  $F_2$ , by  $F_1 \cup F_2$  denote a vertex-disjoint union of  $F_1$  and  $F_2$ . If  $F_1 = F_2 = F$  we will sometimes write  $2F$  instead of  $F \cup F$ .

**Theorem 7** ([10]).

$$\text{ex}^{(1)}(n; P) = \begin{cases} \binom{n}{3} & \text{and } \text{Ex}^{(1)}(n; P) = \{K_n\} & \text{for } n \leq 6, \\ 20 & \text{and } \text{Ex}^{(1)}(n; P) = \{K_6 \cup K_1\} & \text{for } n = 7, \\ \binom{n-1}{2} & \text{and } \text{Ex}^{(1)}(n; P) = \{S_n\} & \text{for } n \geq 8. \end{cases}$$

In [11] we have completely determined the second order Turán number  $\text{ex}^{(2)}(n; P)$ , together with the corresponding 2-extremal 3-graphs. A *comet*  $\text{Co}(n)$  is an  $n$ -vertex 3-graph consisting of the complete 3-graph  $K_4$  and the full star  $S_{n-3}$ , sharing exactly one vertex which is the center of the star (see Fig. 1). This vertex is called *the center* of the comet, while the set of the remaining three vertices of the  $K_4$  is called its *head*.

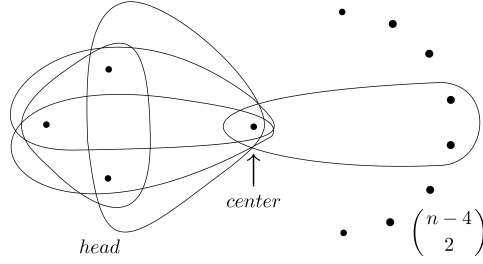


Figure 1: The comet  $\text{Co}(n)$

**Theorem 8** ([11]).

$$\text{ex}^{(2)}(n; P) = \begin{cases} 15 & \text{and } \text{Ex}^{(2)}(n; P) = \{S_7\} & \text{for } n = 7, \\ 20 + \binom{n-6}{3} & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6 \cup K_{n-6}\} & \text{for } 8 \leq n \leq 12, \\ 40 & \text{and } \text{Ex}^{(2)}(n; P) = \{2K_6 \cup K_1, \text{Co}(13)\} & \text{for } n = 13, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}^{(2)}(n; P) = \{\text{Co}(n)\} & \text{for } n \geq 14. \end{cases}$$

In [11] ( $n = 12$ ) and in [15] (for all  $n$ ), we calculated the third order Turán number for  $P$ .

**Theorem 9** ([11],[15]).

$$\text{ex}^{(3)}(n; P) = \begin{cases} 3n - 8 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } 7 \leq n \leq 10, \\ 25 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n), \text{Co}(n)\} & \text{for } n = 11, \\ 32 & \text{and } \text{Ex}^{(3)}(n; P) = \{\text{Co}(n)\} & \text{for } n = 12, \\ 20 + \binom{n-7}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_6 \cup S_{n-6}\} & \text{for } 13 \leq n \leq 14, \\ 4 + \binom{n-5}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_4 \cup S_{n-4}\} & \text{for } n \geq 15. \end{cases}$$

Surprisingly, as an immediate consequence we obtained also an exact formula for the 4th Turán number for  $P$ . Let  $K_5^{+t}$  be the 3-graph obtained from  $K_5$  by fixing two of its vertices, say  $a, b$ , and adding  $t$  more vertices  $v_1, v_2, \dots, v_t$  and  $t$  edges  $\{a, b, v_i\}$ ,  $i = 1, 2, \dots, t$ .

**Theorem 10** ([15]).

$$\text{ex}^{(4)}(n; P) = \begin{cases} 12 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n), K_5^{+2}\} & \text{for } n = 7, \\ 2n - 2 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } 8 \leq n \leq 9, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_5 \cup K_5\} & \text{for } n = 10, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } n = 11, \\ 28 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } n = 12, \\ 33 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_6 \cup G_1(n), K_6 \cup G_2(n)\} & \text{for } n = 13, \\ 40 & \text{and } \text{Ex}^{(4)}(n; P) = \{2K_6 \cup 2K_1, K_4 \cup S_{10}\} & \text{for } n = 14, \\ 48 & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n), K_6 \cup S_9\} & \text{for } n = 15, \\ 3 + \binom{n-5}{2} & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n)\} & \text{for } n \geq 16. \end{cases}$$

The main Turán-type result of this paper provides a complete formula for the fifth order Turán number for  $P$ .

**Theorem 11.**

$$\text{ex}^{(5)}(n; P) = \begin{cases} 11 & \text{and} & \text{Ex}^{(5)}(n; P) = \text{Ex}^{(4)}(7; M) & \text{for } n = 7, \\ 13 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5^{+3}\} & \text{for } n = 8, \\ 14 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5^{+4}, K_5 \cup K_4\} \cup \text{Ex}(9; \{P, C\}|M) & \text{for } n = 9, \\ 19 & \text{and} & \text{Ex}^{(5)}(n; P) = \{\text{Co}(10)\} & \text{for } n = 10, \\ 19 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_4 \cup S_7\} & \text{for } n = 11, \\ 25 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_7, K_4 \cup S_8\} & \text{for } n = 12, \\ 32 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_4 \cup S_9, K_6 \cup K_5^{+2}, K_6 \cup G_3(7)\} & \text{for } n = 13, \\ 39 & \text{and} & \text{Ex}^{(5)}(n; P) = \{\text{Ro}(14)\} & \text{for } n = 14, \\ 46 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{10}\} & \text{for } n = 15, \\ 56 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_6 \cup S_{10}\} & \text{for } n = 16, \\ 65 & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{12}, K_6 \cup S_{11}\} & \text{for } n = 17, \\ 10 + \binom{n-6}{2} & \text{and} & \text{Ex}^{(5)}(n; P) = \{K_5 \cup S_{n-5}\} & \text{for } n \geq 18. \end{cases}$$

To determine Turán numbers, it is sometimes useful to rely on Theorem 3 and divide all 3-graphs into those which contain  $M$  and those which do not. To this end, it is convenient to define conditional Turán numbers (see [10, 11]). For a family of 3-graphs  $\mathcal{F}$ , an  $\mathcal{F}$ -free 3-graph  $G$ , and an integer  $n \geq |V(G)|$ , the *conditional Turán number* is defined as

$$\text{ex}(n; \mathcal{F}|G) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } H \supseteq G\}$$

Every  $n$ -vertex  $\mathcal{F}$ -free 3-graph with  $\text{ex}(n; \mathcal{F}|G)$  edges and such that  $H \supseteq G$  is called  *$G$ -extremal for  $\mathcal{F}$* . We denote by  $\text{Ex}(n; \mathcal{F}|G)$  the family of all  $n$ -vertex 3-graphs which are  $G$ -extremal for  $\mathcal{F}$ . (If  $\mathcal{F} = \{F\}$ , we simply write  $F$  instead of  $\{F\}$ .)

To illustrate the above mentioned technique, observe that for  $n \geq 7$

$$\text{ex}^{(2)}(n; P) = \max\{\text{ex}(n; P|M), \text{ex}^{(2)}(n; M)\} \stackrel{\text{Thm 3}}{=} \max\{\text{ex}(n; P|M), 3n-8\} = \text{ex}(n; P|M),$$

the last equality holding for sufficiently large  $n$  (see [11] for details).

In the proof of Theorem 11 we will use the following five lemmas, all proved in [11] and [14]. For the first two we need one more piece of notation. If, in the above definition, we restrict ourselves to connected 3-graphs only (connected in the weakest, obvious sense) then the corresponding conditional Turán number and the extremal family are denoted by  $\text{ex}_{\text{conn}}(n; \mathcal{F}|G)$  and  $\text{Ex}_{\text{conn}}(n; \mathcal{F}|G)$ , respectively.

**Lemma 1** ([11]). *For  $n \geq 7$ ,*

$$\text{ex}_{\text{conn}}(n; P|C) = 3n - 8 \text{ and } \text{Ex}_{\text{conn}}(n; P|C) = \{G_1(n), G_2(n)\}.$$

Lemma 1 as stated in [11] does not provide family  $\text{Ex}_{\text{conn}}(n; P|C)$ . However, it is clear from its proof that the  $C$ -extremal 3-graphs are the same as in Theorem 3. We will need also another lemma, which is not stated explicitly in [11], but it immediate results from the proof of the previous one.

**Lemma 2** ([11]). *For  $n \geq 7$ ,*

$$\text{ex}_{\text{conn}}(n; P|\{C, M\}) = n + 5 \text{ and } \text{Ex}_{\text{conn}}(n; P|\{C, M\}) = \{K_5^{+(n-5)}\}.$$

*Moreover, if  $H$  is  $n$ -vertex connected  $P$ -free 3-graph such that  $C \subset H$  and  $M \subset H$ , then  $H \subseteq K_5^{+(n-5)}$*

**Lemma 3** ([11]).

$$\text{ex}(n; \{P, C\}|M) = \begin{cases} 2n - 4 & \text{for } 6 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}(n; \{P, C\}|M) = \{\text{Co}(n)\} \text{ for } n \geq 11. \end{cases}$$

**Lemma 4** ([11]). *For  $n \geq 6$*

$$\text{ex}(n; \{P, C, P_2 \cup K_3\}|M) = 2n - 4,$$

*where  $P_2$  is a pair of edges sharing one vertex.*

**Lemma 5** ([14]). *For  $n \geq 6$ ,*

$$\text{ex}^{(2)}(n; \{M, C\}) = \max\{10, n\}.$$

### 3 Proof of Theorem 2

As mentioned in the Introduction, Jackowska has shown in [9], that  $R(P; r) \geq r + 6$  for all  $r \geq 1$ . We are going to show that  $R(P; 10) \leq 16$ .

We will show that every 10-coloring of  $K_{16}$  yields a monochromatic copy of  $P$ . The idea of the proof is to gradually reduce the number of vertices and colors (by one in each step), until we reach a coloring which yields a monochromatic copy of  $P$ .

Let us consider an arbitrary 10-coloring of  $K_{16}$ ,  $K_{16} = \bigcup_{i=1}^{10} G_i$ , and assume that for each  $i \in [10]$ ,  $P \not\subseteq G_i$ . Since  $|K_{16}| = 560$ , the average number of edges per color is 56, and therefore, by Theorems 7–11, either for each  $i \in [10]$ ,  $G_i = K_6 \cup S_{10}$ , or there exists a color, say  $G_{10}$ , contained in one of the 3-graphs:  $S_{16}$ ,  $\text{Co}(16)$ ,  $K_4 \cup S_{12}$ ,  $\text{Ro}(16)$ . We will show, that the later case must occur. Indeed, for each vertex  $v \in V(K_{16})$  we have  $\deg_{K_{16}}(v) = \binom{15}{2} = 105$  whereas for  $v \in V(K_6 \cup S_{10})$ ,  $\deg_{K_6 \cup S_{10}}(v) \in \{10, 36, 8\}$  depending on whether  $v$  is a vertex of  $K_6$ , the center of the star  $S_{10}$  or an other vertex. Since we are not able to obtain an odd number as a sum of even numbers, we can not decompose  $K_{16}$  into edge-disjoint copies of  $K_6 \cup S_{10}$ . Let us turn back to  $G_{10}$ . No matter in which of the four 3-graph  $G_{10}$  is contained, we remove the center of the star (or comet, or rocket) together with up to four more edges of  $G_{10}$ , so that we get rid of color 10 completely (note that some other colors can also be affected by this deletion).

As a result, we obtain a 3-graph  $H_{15}$  on 15 vertices, colored with 9 colors,  $H_{15} = \bigcup_{i=1}^9 G_i$ , with  $|H(15)| \geq 451$  (with some abuse of notation we will keep denoting the subgraphs of  $G_i$  obtained in each step again by  $G_i$ ). The average number of edges per

color is at least 50.1, and therefore there exists a color, say  $G_9$ , with  $|G_9| \geq 51$ . This time we use Theorems 7–9 to conclude that either  $G_9 \subset S_{15}$  or  $G_9 \subset \text{Co}(15)$ . In either case we remove the center and, in case of the comet, one more edge being its head.

We get a 3-graph  $H(14)$  on 14 vertices with  $|H(14)| \geq 359$ , colored by 8 colors,  $H(14) = \bigcup_{i=1}^8 G_i$ . The average number of edges per color is at least 44.9, and hence there exists a color, say  $G_8$ , with  $|G_8| \geq 45$ . Similarly as in the previous step we reduce the picture to a 3-graph  $H(13)$  on 13 vertices with  $|H(13)| \geq 280$ , colored by 7 colors,  $H(13) = \bigcup_{i=1}^7 G_i$ .

This time the average number of edges per color is at least 40, and therefore, by Theorems 7 and 8, either each color is a copy of  $\text{Co}(13)$  or  $K_6 \cup K_6 \cup K_1$ , or there exists a color, say  $G_7$ , contained in the full star  $S_{13}$ . We will show in the similar way as before, that  $H(13)$  can not be decomposed into edge-disjoint copies of  $\text{Co}(13)$  and  $K_6 \cup K_6 \cup K_1$ , and therefore the later case must occur. Indeed, first notice that there is not enough space for two edge-disjoint copies of  $K_6 \cup K_6 \cup K_1$  in  $K_{13}$  and therefore also in  $H(13)$ . Fixed one copy of  $K_6 \cup K_6 \cup K_1$  in  $K_{13}$ . By pigeon-hole principle, any other copy of  $K_6$  must share at least three vertices with one of the fixed copies of  $K_6$  and therefore they are not edge-disjoint. Now observe, that since during our procedure we have lost at most 6 edges of  $K_{13}$ , for each vertex  $v \in V(H(13))$  we have  $\deg_{H(13)}(v) \geq \binom{12}{2} - 6 = 60$  and also for each vertex of a comet  $\text{Co}(13)$  which is not its center, we have  $\deg_{\text{Co}(13)}(v) \leq 8$ . Since we can decompose  $H(13)$  into at most seven copies of  $\text{Co}(13)$ , there must exist a vertex  $v \in V(H(13))$  which is not a center of any of these comets and therefore  $\deg_{H(13)}(v) \leq 10 + 6 \cdot 8 = 58 < 60$ , a contradiction. Consequently we have  $G_7 \subseteq S_{13}$  and, by removing the center of this star, we obtain a 6-coloring of a 3-graph  $H(12)$  on 12 vertices with  $|H(12)| \geq 214$ .

To proceed, let us assume for a while, that none of the colors  $G_i$ ,  $i \in [6]$ , is a star. Then, by Theorems 7–9, each color with more than 32 edges is a subset of  $K_6 \cup K_6$ . The average number of edges per color is at least 35.6, and hence there exists a color, say  $G_6$ , with  $G_6 \subset K_6 \cup K_6$ . We remove all edges of this copy of  $K_6 \cup K_6$ , getting a bipartite 3-graph  $H'(12)$  with a bipartition  $V(H'(12)) = V \cup U$ ,  $|V| = |U| = 6$ , and with  $|H'(12)| \geq 174$  edges colored by 5 colors,  $H'(12) = \bigcup_{i=1}^5 G_i$ . Note, that every subgraph of  $K_6 \cup K_6$  contained in  $H'(12)$  (and consequently each color class of  $H'(12)$ ) has at most 36 edges. Since  $3 \cdot 36 + 2 \cdot 32 = 172 < 174$ , at least 3 colors must be subsets of  $K_6 \cup K_6$  and have at least 34 edges. Now observe, that if two color classes, say  $G_1$  and  $G_2$ , have at least 34 edges each, then they are disjoint unions of two copies of  $K_6$ , one of the vertex set  $U'_i \cup W'_i$ , the other one on  $U''_i \cup W''_i$ , with four missing edges  $U'_i, U''_i, W'_i, W''_i$ , where  $U = U'_i \cup U''_i$ ,  $V = V'_i \cup V''_i$ ,  $i = 1, 2$ , and  $\{U'_1, U''_1\} = \{U'_2, U''_2\}$  (See Fig. 2). Otherwise, if  $1 \leq |U'_1 \cap U'_2| \leq 2$ ,  $G_1$  and  $G_2$  would share at least six edges, and thus  $|G_1| + |G_2| \leq 36 + 36 - 6 < 2 \cdot 34$ . This simply means that one of the partitions, of  $U$  or of  $W$ , must be swapped. But this is impossible for three color classes. Consequently, at least one color, say  $G_6$ , is a star. We remove the center of this star to get a 5-coloring of a 3-graph  $H(11)$  on 11 vertices with  $|H(11)| \geq 159$ .

By repeating this argument three more times, we finally arrive at a 2-coloring of a 3-graph  $H(8) = G_1 \cup G_2$ , with  $|H(8)| \geq 50$  which, by Theorem 7, should contain a copy of  $P$ , a contradiction.  $\square$

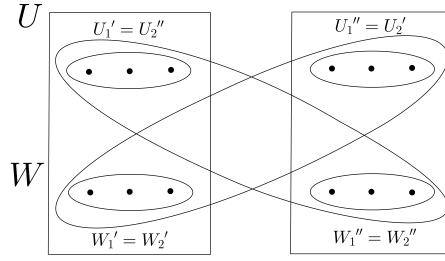


Figure 2: The partition of the set of vertices of  $H'(12)$ ,  $G_1$  and  $G_2$ .

## 4 Proof of Theorem 11

Let us define  $\mathcal{H}_n = \text{Ex}^{(1)}(n; P) \cup \text{Ex}^{(2)}(n; P) \cup \text{Ex}^{(3)}(n; P) \cup \text{Ex}^{(4)}(n; P)$ . To prove Theorem 11 we need to find, for each  $n \geq 7$ , a  $P$ -free,  $n$ -vertex 3-graph  $H$  with the biggest possible number of edges such that, whenever  $G \in \mathcal{H}_n$  then  $H \not\subseteq G$ . Moreover we will show, that  $|H| = h_n$ , where  $h_n$  is the number of edges, given by the formula to be proved.

First note that for each  $n \geq 7$ , all candidates for being 5-extremal 3-graphs do qualify, that is, are  $P$ -free, are not contained in any of the 3-graphs from  $\mathcal{H}_n$ , and have  $h_n$  edges. To finish the proof, we will show that each  $P$ -free,  $n$ -vertex 3-graph  $H$ , not contained in any of 3-graph from  $\mathcal{H}_n$  satisfy  $|H| < h_n$  unless it is one of the candidates for being 5-extremal 3-graph itself.

For the latter task, we distinguish two cases: when  $H$  is connected and disconnected. The entire proof is inductive, in the sense that here and there we apply the very Theorem 11 for smaller instances of  $n$ , once they have been confirmed.

Let for all  $n \geq 7$ ,  $H$  be  $P$ -free  $n$ -vertex 3-graph such that for each  $G \in \mathcal{H}_n$ ,  $H \not\subseteq G$ . Moreover let  $H$  be different from all candidates for being 5-extremal 3-graphs with the same number of vertices. We will show that  $|H| < h_n$ .

### 4.1 Connected case

We start with the connected case. First let us assume, that  $M \not\subseteq H$  and consider consecutive intersecting families. Recall that for all  $n \geq 7$ ,  $H \not\subseteq S_n$ , for  $7 \leq n \leq 12$ ,  $H \not\subseteq G_1(n)$  and  $H \not\subseteq G_2(n)$ , for  $7 \leq n \leq 9$  and  $n = 11$ ,  $H \not\subseteq G_3(n)$  and finally, for  $n = 7$   $H$  is not equal to any of 4-extremal 3-graphs for  $M$ . Therefore, by Theorems 3, 4 and 5, we get that for all  $n \geq 7$ ,

$$|H| < h_n.$$

Consequently we will be assuming by the end of the proof, that  $M \subset H$ . If additionally  $C \subset H$ , then by Lemma 2,  $H \subseteq K_5^{+(n-5)}$  and hence  $|H| \leq |K_5^{+(n-5)}| = n + 5$ . Therefore, for  $n \geq 10$ ,  $|H| < h_n$ . If  $n = 7$ , as  $K_5^{+2} \in \mathcal{H}_7$ , we have  $H \not\subseteq K_5^{+2}$  and thus we may exclude this case. Lastly, for  $8 \leq n \leq 9$ , by the definition of  $H$ ,  $H \neq K_5^{+(n-5)}$  and hence  $|H| < h_n$ . Therefore, in the rest of the proof we will be assuming, that  $C \not\subseteq H$ .



Finally, let  $H$  be connected  $\{P, C\}$ -free 3-graph containing  $M$ . Then by Lemma 3, for  $7 \leq n \leq 8$ ,  $|H| \leq 2n - 4 < h_n$  and for  $n = 9$ , since  $H \notin \text{Ex}(9, \{P, C\}|M)$ , we have  $|H| < 14 = h_9$ .

For  $10 \leq n \leq 11$  we need two more facts, which we state here without the proof. Namely  $\text{ex}_{\text{conn}}(10; \{P, C\}|M) = 19$  and  $\text{Ex}_{\text{conn}}(10; \{P, C\}|M) = \{\text{Co}(10)\}$ . Since, by the definition of  $H$ ,  $H \neq \text{Co}(10)$ , this implies, that  $|H| < 19 = h_{10}$ . Whereas for  $n = 11$  we have  $\text{ex}_{\text{conn}}^{(2)}(11; \{P, C\}|M) = 18$ , and therefore, as  $H \not\subseteq \text{Co}(11)$ , we get  $|H| \leq \text{ex}_{\text{conn}}^{(2)}(11; \{P, C\}|M) = 18 < 19 = h_{11}$ .

Recall, that for all  $n \geq 11$ ,  $H \not\subseteq \text{Co}(n)$ . Moreover, for  $12 \leq n \leq 13$ , since  $|\text{Ro}(n)| < h_n$ , we may assume, that  $H \not\subseteq \text{Ro}(n)$ . Further, for  $n = 14$ , by the definition of  $H$  we have  $H \neq \text{Ro}(14)$  and thus, if  $H \subset \text{Ro}(14)$ , then  $|H| < |\text{Ro}(14)| = h_n$ . Finally for all  $n \geq 15$  we have  $H \not\subseteq \text{Ro}(n)$ . Therefore, since for all  $n \geq 12$  we have

$$h_n \leq \binom{n-6}{2} + 10,$$

to complete the proof of the connected case it is enough to prove the following Lemma,

**Lemma 6.** *If  $H$  is a connected,  $n$ -vertex,  $n \geq 12$ ,  $\{P, C\}$ -free 3-graph containing  $M$  such that  $H \not\subseteq \text{Co}(n)$  and  $H \not\subseteq \text{Ro}(n)$ , then  $|H| < \binom{n-6}{2} + 10$ .*

We devote an entire Section 5 to prove Lemma 6.

## 4.2 Disconnected case

Now let  $H$  be disconnected and let  $m = m(H)$  be the number of vertices in the smallest component of  $H$ . We have  $m \neq 2$ , since no component of a 3-graph may have two vertices. We now break the proof into several cases.

Let us express  $H$  as a vertex disjoint union of two 3-graphs:

$$H = H' \cup H'', \quad |V(H')| = m, \quad |V(H'')| = n - m$$

Then, clearly, both  $H'$  and  $H''$  are  $P$ -free, and thus

$$|H| \leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P). \quad (2)$$

Below, to bound  $|H|$ , we use the Turán numbers for  $P$  of the 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup> and 5<sup>th</sup> order and utilize, respectively, Theorems 7, 8, 9, 10 and 11 (per induction).

Let  $v$  be an isolated vertex ( $\mathbf{m} = \mathbf{1}$ ). Since for  $n = 7$  and any 3-graph  $H''$ ,  $K_1 \cup H'' \subseteq K_1 \cup K_6 \in \mathcal{H}_7$ , we may assume that  $n \geq 8$ . For  $8 \leq n \leq 11$ , as  $H$  cannot be a sub-3-graph of  $S_n$ ,  $K_6 \cup K_{n-6}$ ,  $G_1(n)$  or  $G_2(n)$ ,  $H''$  is not a sub-3-graph of  $S_{n-1}$ ,  $K_6 \cup K_{n-7}$ ,  $G_1(n-1)$  and  $G_2(n-1)$ . Consequently, for  $n = 8, 10$ ,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$

For  $n = 9$  additionally we have  $H'' \not\subseteq G_3(8)$  and therefore

$$|H| \leq \text{ex}^{(5)}(8; P) = 13 < 14 = h_9,$$

whereas for  $n = 11$ ,  $H'' \not\subseteq K_5 \cup K_5$  and  $H'' \not\subseteq \text{Co}(10)$ . Consequently

$$|H| = |H''| < \text{ex}^{(5)}(10; P) = 19 = h_{11}.$$

For  $n \geq 12$ , since  $H = K_1 \cup H''$  is not a sub-3-graph of any of the 3-graphs in  $\mathcal{H}_n$ , we have  $H'' \not\subseteq S_{n-1}$  and  $H'' \not\subseteq \text{Co}(n-1)$ . Moreover, for  $n = 12, 13$ ,  $H'' \not\subseteq K_6 \cup K_{n-7}$ , for  $n = 12$ ,  $H'' \not\subseteq G_1(n-1)$  and  $H'' \not\subseteq G_2(n-1)$ , for  $n = 14$ ,  $H'' \not\subseteq 2K_6 \cup K_1$ , for  $n = 14, 15$ ,  $H'' \not\subseteq K_6 \cup S_{n-7}$  and finally, for  $n \geq 15$ ,  $H'' \not\subseteq K_4 \cup S_{n-5}$ . Consequently,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$

For  $\mathbf{m} = 3$  and  $n = 7, 8$ , by (2) we get

$$|H| \leq \text{ex}^{(1)}(3; P) + \text{ex}^{(1)}(n-3; P) = 1 + \text{ex}^{(1)}(n-3; P) < h_n,$$

Since each disconnected 3-graph  $H = H' \cup H''$  with  $|V(H')| = 3$  and  $|V(H'')| = 6$  is a sub-3-graph of  $K_3 \cup K_6 \in \mathcal{H}_9$ , we may assume that  $n \neq 9$ . For  $n = 10$  we have  $K_3 \cup K_6 \cup K_1 \subset K_4 \cup K_6 \in \mathcal{H}_{10}$ . Consequently  $H'' \not\subseteq K_6 \cup K_1$  and thus  $|H''| \leq \text{ex}^{(2)}(7; P) = 15$ . Hence  $|H| \leq 1 + 15 = 16 < 19 = h_{10}$ .

Further, for all  $n \geq 11$ , since  $\text{Co}(n) \in \mathcal{H}_n$ , we have  $H'' \not\subseteq S_{n-3}$ . Therefore for  $n \geq 12$ ,

$$|H| \leq 1 + \text{ex}^{(2)}(n-3; P) < h_n,$$

whereas, for  $n = 11$  additionally we have  $H \not\subseteq K_3 \cup K_6 \cup K_2 \subset K_6 \cup K_5 \in \mathcal{H}_{11}$ . Thus  $H'' \not\subseteq K_6 \cup K_2$  and consequently

$$|H| \leq 1 + \text{ex}^{(3)}(8; P) = 17 < 19 = h_{11}.$$

For  $\mathbf{m} = 4$  and  $n = 8$  by (2) we have

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(1)}(4; P) = 4 + 4 = 8 < h_8.$$

For  $n = 9$ , by the definition of  $H$ ,  $H \neq K_4 \cup K_5$  and therefore  $|H| < |K_4 \cup K_5| = 14 = h_9$ . Similarly like before, we may skip the case  $n = 10$ , because each disconnected 3-graph  $H = H' \cup H''$  with  $|V(H')| = 4$  and  $|V(H'')| = 6$  is a sub-3-graph of  $K_4 \cup K_6 \in \mathcal{H}_{10}$ . For  $n = 11$ , since  $K_4 \cup K_6 \cup K_1 \subset K_5 \cup K_6 \in \mathcal{H}_{11}$ , we have  $H'' \not\subseteq K_6 \cup K_1$  and therefore  $|H''| \leq \text{ex}^{(2)}(7; P) = 15$  with the equality only for  $H'' = S_7$ . But, by the definition of  $H$ ,  $H \neq K_4 \cup S_7$ , and hence

$$|H| < |K_4 \cup S_7| = 19 = h_{11}.$$

Further, for  $n = 12, 13$ , since  $\text{Ex}^{(1)}(n-4; P) = \{S_{n-4}\}$  and  $H \neq H_4 \cup S_{n-4}$ , we have  $|H| < |H_4 \cup S_{n-4}| = h_n$ . Finally, for  $n \geq 14$ , since  $K_4 \cup S_{n-4} \in \mathcal{H}_n$  we get  $H'' \not\subseteq S_{n-4}$  and consequently,

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(2)}(n-4; P) < h_n.$$

Now let  $\mathbf{m} = 5$ . Notice that each disconnected 3-graph  $H = H' \cup H''$  with  $|V(H')| = 5$  and  $5 \leq |V(H'')| \leq 6$  is a sub-3-graph of  $K_5 \cup K_5 \in \mathcal{H}_{10}$  and  $K_5 \cup K_6 \in \mathcal{H}_{11}$  respectively.

Therefore we may consider only  $n \geq 12$ . For  $n = 12$ , since  $K_5 \cup K_6 \cup K_1 \subset K_6 \cup K_6 \in \mathcal{H}_{12}$ , we have  $|H''| \leq \text{ex}^{(2)}(7; P) = 15$  with the equality only for  $H'' = S_7$ . But, by the definition of  $H$ ,  $H \neq K_5 \cup S_7$  and hence  $|H| < |K_5 \cup S_7| = 25 = h_{12}$ . Finally, for  $n \geq 13$ , by (2),

$$|H| \leq \text{ex}^{(1)}(5; P) + \text{ex}^{(1)}(n - 5; P) = 10 + \binom{n-6}{2} \leq h_n,$$

where the equality is achieved only by the candidates for 5-extremal 3-graphs with the proper number of vertices.

For  $\mathbf{m} = \mathbf{6}$  we have  $n \geq 12$ , but as each disconnected 3-graph  $H' \cup H''$  with  $|V(H')| = |V(H'')| = 6$  is a sub-3-graph of  $K_6 \cup K_6 \in \mathcal{H}_{12}$ , we may consider only  $n \geq 13$ . Recall, that  $\{2K_6 \cup K_1, K_6 \cup S_7, K_6 \cup G_1(7), K_6 \cup G_2(7)\} \subset \mathcal{H}_{13}$  and therefore, for  $n = 13$ ,  $H''$  is not contained in any of the 3-graphs  $K_6 \cup K_1, S_7, G_1(7), G_2(7)$ . Consequently,  $|H''| \leq \text{ex}^{(4)}(7; P) = 12$  with the equality only for  $H'' = G_3(7)$  and  $H'' = K_5^{+2}$ . But, by the definition of  $H$ ,  $H \neq K_6 \cup K_5^{+2}$  and  $H \neq K_6 \cup G_3(7)$  and thus

$$|H| < |K_6 \cup K_5^{+2}| = |K_6 \cup G_3(7)| = h_{13}.$$

For the same reason, if  $n = 14$ , then  $H'' \not\subset S_8$  and  $H'' \not\subset K_6 \cup K_2$ . Consequently,

$$|H| = |H'| + |H''| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(3)}(8; P) = 20 + 16 < 39 = h_{14},$$

whereas for  $n = 15$ , we have  $H'' \not\subset S_9$  and hence

$$|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(2)}(9; P) = 20 + 21 < 46 = h_{15}.$$

Further, for  $n = 16, 17$ , by the definition of  $H$ ,  $H \neq K_6 \cup S_{n-6}$ . Consequently, as  $\text{Ex}(n-6; P) = \{S_{n-6}\}$ , we get

$$|H| < |K_6 \cup S_{n-6}| = h_n.$$

Finally, for  $n \geq 18$ , by (2),

$$|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(1)}(n-6; P) = 20 + \binom{n-7}{2} < \binom{n-6}{2} + 10 = h_n.$$

If  $\mathbf{m} = \mathbf{7}$ , then  $n \geq 14$ . For  $n = 14$ , since  $H \not\subset 2K_6 \cup 2K_1 \in \mathcal{H}_{14}$ , at least one of the components of  $H$  is not a sub-3-graph of  $K_6 \cup K_1$  and therefore has at most  $\text{ex}^{(2)}(7; P) = 15$  edges. Consequently,

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(2)}(7; P) = 20 + 15 < 39 = h_{14}.$$

To bound the number of edges of  $H$  for  $n \geq 15$  we use (2) to get

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(1)}(n-7; P) = 20 + \binom{n-8}{2} < \binom{n-6}{2} + 10 \leq h_n.$$

Finally, for  $\mathbf{m} \geq \mathbf{8}$  we have  $n \geq 16$  and, by (2),

$$\begin{aligned} |H| &\leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n-m; P) = \binom{m-1}{2} + \binom{n-m-1}{2} \\ &\leq \binom{7}{2} + \binom{n-9}{2} < \binom{n-6}{2} + 10 \leq h_n. \end{aligned}$$

## 5 The proof of Lemma 6

Recall that  $H$  is a connected,  $n$ -vertex,  $n \geq 12$ ,  $\{P, C\}$ -free 3-graph such that  $M \subset H$ ,  $H \not\subseteq \text{Co}(n)$  and  $H \not\subseteq \text{Ro}(n)$ . We need to show that

$$|H| < \binom{n-6}{2} + 10.$$

Since for  $n \geq 11$ , by Lemma 4

$$\text{ex}(\{n; P, C, P_2 \cup K_3\} | M) = 2n - 4 < \binom{n-6}{2} + 10,$$

we may assume that  $P_2 \cup K_3 \subset H$ . Let us denote a copy of  $P_2$  from  $P_2 \cup K_3$  in  $H$  by  $Q$  and the vertex of degree two in  $Q$  by  $x$ . We let  $U = V(Q)$ ,  $V = V(H)$  and  $W = V \setminus U$ . Moreover, let  $W_0$  be the set of vertices of degree zero in  $H[W]$  and  $W_1 = W \setminus W_0$ . (see Fig. 3). Note that, by definition,  $H[W] = H[W_1]$  and  $|W_1| \geq 3$ .

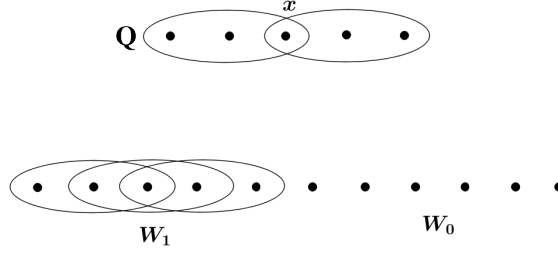


Figure 3: Set-up for the proof of Lemma 6

We also split the set of edges of  $H$ . First, notice that, since  $H$  is  $P$ -free, there is no edge with one vertex in each  $U$ ,  $W_0$ , and  $W_1$ . We define  $H_i = \{h \in H : h \cap U \neq \emptyset, h \cap W_i \neq \emptyset\}$ , where  $i = 0, 1$ . Then, clearly,

$$H = H[U] \cup H[W] \cup H_0 \cup H_1, \quad (3)$$

with all four parts edge-disjoint. Since by definition  $H[U] \cup H_0 = H[U \cup W_0]$ , sometimes we will use the following equality

$$H = H[U \cup W_0] \cup H_1 \cup H[W]. \quad (4)$$

Recall that  $H$  is  $C$ -free, and therefore one can use Theorem 6 to get the bounds, for  $|W_0| \geq 1$

$$|H[U \cup W_0]| \leq \binom{|U \cup W_0| - 1}{2} = \binom{|W_0| + 4}{2} \quad (5)$$

and for  $|W_1| \geq 6$ ,

$$|H[W]| \leq \binom{|W_1| - 1}{2} \quad (6)$$

Notice that for each edge  $h \in H_0 \cup H_1$  with  $|h \cap U| = 1$  we have  $h \cap U = \{x\}$ , because otherwise  $h$  together with  $Q$  would form a copy of  $P$  in  $H$ . We let

$$F^0 = \{h \in H_0 \cup H_1 : h \cap U = \{x\}\}.$$

Also, to avoid a copy of  $C$  in  $H$ , if for  $h \in H_0 \cup H_1$  we have  $|h \cap U| = 2$  then the pair  $h \cap U$  is contained in an edge of  $Q$ . For  $k = 1, 2$ , we define

$$F^k = \{h \in H_0 \cup H_1 : |h \cap U \setminus \{x\}| = k\}.$$

Clearly,  $H_0 \cup H_1 = F^0 \cup F^1 \cup F^2$  (see Fig. 4). Further, for  $i = 0, 1$  and  $k = 0, 1, 2$ , we set

$$F_i^k = F^k \cap H_i.$$

It was noticed in [11] that, as  $H$  is  $P$ -free,  $F_1^1 = \emptyset$  and therefore,

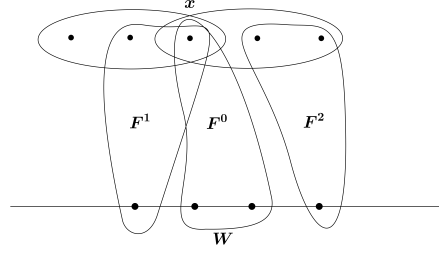


Figure 4: Three types of edges in  $H_0 \cup H_1$

$$H_1 = F_1^0 \cup F_1^2. \quad (7)$$

Moreover, for all  $v \in W$  we have

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset, \quad (8)$$

and, by the definition of  $F^1$  and  $F^2$ ,

$$|F^1(v)| \leq 4 \quad \text{and} \quad |F^2(v)| \leq 2. \quad (9)$$

where for a given subset of edges  $G \subseteq H$  and for a vertex  $v \in V(H)$  we set  $G(v) = \{h \in G : v \in h\}$ .

In the whole proof we will be using the fact, that for all edges  $e \in F^0$ , the pair  $e \cap W_1$  is *nonseparable* in  $H[W]$ , that is, every edge of  $H[W]$  must contain both these vertices or none. Consequently, for each  $v \in W_0$ ,  $|F^0(v)| \leq |W_0| - 1$  and thus, by (8) and (9),

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leq 4 + \max\{2, |W_0| - 1\}. \quad (10)$$

Observe also that, because  $H$  is connected,  $H_1 \neq \emptyset$ . Consequently, since the presence of any edge of  $H_1$  forbids at least 4 edges of  $H[U]$ ,

$$|H[U]| \leq 6. \quad (11)$$

Moreover, in [11] the authors have proved the following bounds on the number of edges in  $H_1$ :

$$\text{For } |W_1| \geq 4, \quad |F_1^2| \leq 2|W_1| - 4. \quad (12)$$

$$\text{For } |W_1| \geq 3, \quad |H_1| \leq 2|W_1| - 3. \quad (13)$$

$$\text{For } |W_1| \geq 3, \quad |F_1^0| \leq |W_1|. \quad (14)$$

As a consequence of these inequalities one can prove the following

$$\text{For } |W_1| \geq 7, \quad |H[U]| + |H_1| \leq 2|W_1| - 1. \quad (15)$$

Indeed, if  $|H_1| \leq |W_1|$ , then (15) results from (11) and the inequality  $|W_1| - 1 \geq 7 - 1 = 6$ . Otherwise, by (14), (7) and (8), there exists a vertex  $v \in W_1$ , such that  $|F_1^2(v)| = 2$ . As  $H$  is  $\{P, C\}$ -free, by the definition of  $F_1^2(v)$ , this implies, that  $|H[U]| = 2$  and (15) follows from (13).

We also need the following fact proven in [15].

**Fact 1.** [15] If  $F_1^2 \neq \emptyset$ , then

$$|H[U \cup W_0]| \leq \begin{cases} 8 & \text{for } |W_0| = 1, \\ 3|W_0| + 7 & \text{for } 2 \leq |W_0| \leq 4, \\ \binom{|W_0|+2}{2} + 1 & \text{for } |W_0| \geq 5. \end{cases} \quad (16)$$

We split the whole proof of Lemma 6 into a few short parts, Facts 2-6.

**Fact 2.** For  $n \geq 13$  if  $W_0 = \emptyset$  and  $H_1 \neq \emptyset$ , then  $|H| < 10 + \binom{n-6}{2}$ .

*Proof.* Let us consider two cases, whether or not  $H[W] \subseteq S_{n-5}$ . If  $H[W] \subseteq S_{n-5}$  then, since  $H$  is  $P$ -free, by (9),  $|F^2| = |F^2(y)| \leq 2$  where  $y \in W_1$  is the center of the star  $S_{n-5}$ . Additionally if  $F_1^0 = \emptyset$ , then by (3), (11), (7) and (6),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + \binom{n-6}{2} = \binom{n-6}{2} + 8 < \binom{n-6}{2} + 10.$$

Otherwise  $F_1^0 \neq \emptyset$ . As for each  $h \in F_1^0$ , the pair  $h \cap W_0$  is nonseparable, one can show, that  $|H[W]| \leq \binom{n-8}{2} + 1$ . By (14),  $|F_1^0| \leq |W_1| = n - 5$  and hence by (7),  $|H_1| \leq n - 5 + 2 = n - 3$ . Consequently, by (3) and (11),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + n - 3 + \binom{n-8}{2} + 1 = \binom{n-7}{2} + 12 < \binom{n-6}{2} + 10.$$

Now we move to the case  $H[W] \not\subseteq S_{n-5}$  and use Theorem 7 to bound the number of edges in  $H[W]$  by  $\text{ex}^{(2)}(n-5; P)$ . Moreover, by (15),  $|H[U]| + |H_1| \leq 2(n-5) - 1 = 2n - 11$ . Consequently, by (3) and Theorem 8,

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 2 \cdot n - 11 + \text{ex}^{(2)}(n-5; P) < \binom{n-6}{2} + 10,$$

where the last inequality is valid for  $n \geq 14$ . For  $n = 13$  we have to strengthen the bound of  $H[W]$ . As  $H[W] \not\subseteq K_6 \cup K_2$ ,  $H[W] \neq G_1(8)$  and  $H[W] \neq G_2(8)$ , by Theorems 7, 8 and 9 we have  $|H[W]| < \text{ex}^{(3)}(8; P) = 16$  and therefore

$$|H| < 2 \cdot 13 - 11 + 16 = 31 = \binom{13-6}{2} + 10.$$

□

**Fact 3.** For  $n \geq 13$  if  $H_1 \neq \emptyset$ ,  $H \not\subseteq \text{Co}(n)$  and  $|W_1| = 3$  then  $|H| < 10 + \binom{n-6}{2}$

*Proof.* We have  $|H[W]| = 1$ ,  $|U \cup W_0| = n - 3$  and by (13),  $|H_1| \leq 3$ . Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Consequently all we need to do is to bound the number of edges in  $H[U \cup W_0]$ . Since  $H \not\subseteq \text{Co}(n)$ , either  $F_1^2 \neq \emptyset$  or  $H[U \cup W_0] \not\subseteq S_{n-3}$ . In the former case we use Fact 1 to get  $|H[U \cup W_0]| \leq \binom{n-6}{2} + 1$  and therefore

$$|H| \leq \binom{n-6}{2} + 1 + 4 = \binom{n-6}{2} + 5 < \binom{n-6}{2} + 10.$$

Otherwise  $H[U \cup W_0] \not\subseteq S_{n-3}$ , so by Theorem 7,  $|H[U \cup W_0]| \leq \text{ex}^{(2)}(n-3; P)$ . Consequently, by Theorem 8, for  $13 \leq n \leq 15$ ,  $|H[U \cup W_0]| \leq 20 + \binom{n-3-6}{3}$  and therefore,

$$|H| \leq 20 + \binom{n-9}{3} + 4 = \binom{n-9}{3} + 24 < \binom{n-6}{2} + 10,$$

Whereas for  $n \geq 16$  we get  $|H[U \cup W_0]| \leq 4 + \binom{n-3-4}{2}$ , and hence

$$|H| \leq \binom{n-7}{2} + 4 + 4 = \binom{n-7}{2} + 8 < \binom{n-6}{2} + 10.$$

□

**Fact 4.** For  $n \geq 13$ , if  $H_1 \neq \emptyset$ ,  $H \not\subseteq \text{Ro}(n)$  and  $|W_1| = 4$  then  $|H| < 10 + \binom{n-6}{2}$

*Proof.* The proof goes along the lines of the previous one. We have  $|H[W]| \leq \binom{4}{3} = 4$ ,  $|U \cup W_0| = n - 4$  and by (13),  $|H_1| \leq 5$ . Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 5 + 4 = |H[U \cup W_0]| + 9.$$

Consequently to finish the proof we need to bound  $|H[U \cup W_0]|$ . Since  $H \not\subseteq \text{Ro}(n)$ , either  $F_1^2 \neq \emptyset$  or  $H[U \cup W_0] \not\subseteq S_{n-4}$ . In the former case we use Fact 1 to get for  $n = 13$ ,  $|H[U \cup W_0]| \leq 19$  and consequently

$$|H| \leq 19 + 9 = 28 < 31 = 10 + \binom{13-6}{2}.$$

Whereas for  $n \geq 14$ ,  $|H[U \cup W_0]| \leq \binom{n-7}{2} + 1$  and hence,

$$|H| \leq \binom{n-7}{2} + 1 + 9 = \binom{n-7}{2} + 10 < \binom{n-6}{2} + 10.$$

Otherwise  $H[U \cup W_0] \not\subseteq S_{n-3}$  so we use Theorem 7 to get  $|H[U \cup W_0]| \leq \text{ex}^{(2)}(n-4; P)$ . Consequently, by Theorem 8, for  $13 \leq n \leq 16$ ,  $|H[U \cup W_0]| \leq 20 + \binom{n-4-6}{3}$  and hence

$$|H| \leq 20 + \binom{n-10}{3} + 9 = \binom{n-10}{3} + 29 < \binom{n-6}{2} + 10.$$

Whereas for  $n \geq 17$  we have  $|H[U \cup W_0]| \leq 4 + \binom{n-4-4}{2}$  and therefore,

$$|H| \leq 4 + \binom{n-8}{2} + 9 = \binom{n-8}{2} + 13 < \binom{n-6}{2} + 10.$$

□

**Fact 5.** If  $n = 12$ ,  $H_1 \neq \emptyset$  and  $H \neq \text{Co}(12)$  then  $|H| < 10 + \binom{12-6}{2} = 25$ .

*Proof.* Let us split the proof into five parts according to the size of the set  $W_1$ . We start with  $|\mathbf{W}_1| = \mathbf{3}$ . Then  $|W_0| = 4$ ,  $|U \cup W_0| = 9$ ,  $|H[W]| = 1$  and by (13),  $|H_1| \leq 3$ . Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Further, as  $H \not\subseteq \text{Co}(12)$ , either  $F_1^2 \neq \emptyset$  or  $H[U \cup W_0] \not\subseteq S_{n-3}$ . In the former case we use Fact 1 to get  $|H[U \cup W_0]| \leq 19$ . Otherwise,  $H[U \cup W_0] \not\subseteq S_{n-3}$ , and since  $H[U \cup W_0] \neq K_6 \cup K_3$ , by Theorems 7 and 8,  $|H[U \cup W_0]| < 21$ . In both cases  $|H[U \cup W_0]| \leq 20$  and therefore

$$|H| \leq |H[U \cup W_0]| + 4 \leq 20 + 4 = 24 < 25.$$



For  $|\mathbf{W}_1| = 4$  we have  $|W_0| = 3$ ,  $|U \cup W_0| = 8$  and  $|H[W]| \leq \binom{4}{3} = 4$ . If  $F_1^2 = \emptyset$ , then  $H_1 = F_1^0 \neq \emptyset$  and as for each  $h \in F_1^0$  the pair  $h \cap W_1$  is nonseparable,  $|H_1| = 1$  and  $|H[W]| = 2$ . Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{7}{2} + 1 + 2 = 24 < 25.$$

Otherwise  $F_1^2 \neq \emptyset$  and we can use Fact 1 to get  $|H[U \cup W_0]| \leq 16$ . For  $F_1^0 \neq \emptyset$ ,  $|H[W]| = 2$  and consequently, by (4) and (13),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 5 + 2 = 23 < 25.$$

Whereas for  $F_1^0 = \emptyset$  we use (4), (7) and (12) to get

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 4 + 4 = 24 < 25.$$

Now let  $|\mathbf{W}_1| = 5$ ,  $|W_0| = 2$ ,  $|U \cup W_0| = 7$  and  $|H[W]| \leq \binom{5}{3} = 10$ . For  $F_1^2 \neq \emptyset$ , by Fact 1 we get  $|H[U \cup W_0]| \leq 13$  and moreover  $|H[W]| \leq 6$ , because otherwise we wouldn't be able to avoid a path  $P$  in  $H$ . If additionally  $P_2 \subseteq H[W]$  then again by  $P \not\subseteq H$ ,  $|H_1| = |F_1^0| + |F_1^2| \leq 2 + 2 = 4$ . Hence, by (4)

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 4 + 6 = 23 < 25.$$

Otherwise  $P_2 \not\subseteq H[W]$  and consequently one can show, that  $|H[W]| \leq 4$ . Therefore, by (4) and (13),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 7 + 4 = 24 < 25.$$

For  $F_1^2 = \emptyset$  we have  $F_1^0 \neq \emptyset$ . Hence, since for each  $h \in F_1^0$  the pair  $h \cap W_1$  is nonseparable,  $|H[W]| \leq 4$  and  $|H_1| = |F_1^0| \leq 2$ . Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{7-1}{2} + 2 + 4 = 21 < 25.$$

We move to  $|\mathbf{W}_1| = 6$ . Then  $|W_0| = 1$ ,  $|U \cup W_0| = 6$  and by (6),  $|H[W]| \leq \binom{6-1}{2} = 10$ . Let us again start with the case  $F_1^2 \neq \emptyset$ . By (16) we get  $|H[U \cup W_0]| \leq 8$ . If  $P_2 \subseteq H[W]$  then since  $H$  is  $P$ -free,  $|H_1| = |F_1^0| + |F_1^2| \leq 2 + 4 = 6$ . Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 6 + 10 = 24 < 25.$$

Otherwise  $P_2 \not\subseteq H[W]$  and therefore one can show that  $|H[W]| \leq 6$ . By (13),  $|H_1| \leq 9$  and consequently by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 9 + 6 = 23 < 25.$$

For  $F_1^2 = \emptyset$  we have  $F_1^0 \neq \emptyset$ . Hence since for each  $h \in F_1^0$  the pair  $h \cap W_1$  is nonseparable,  $|H[W]| \leq 8$  and by (14),  $|H_1| = |F_1^0| \leq 6$ . Therefore, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 6 + 8 = 24 < 25.$$

Finally,  $|\mathbf{W}_1| = \mathbf{7}$ ,  $W_0 = \emptyset$  and by (6),  $|H[W]| \leq \binom{7-1}{2} = 15$ . If  $H[W] \subseteq S_7$  then as  $H$  is  $P$ -free, by (9),  $|F_1^2| = |F_1^2(y)| \leq 2$ , where  $y \in W_1$  is the center of the star  $S_7$ . If additionally  $F_1^0 = \emptyset$ , then by (3), (7) and (11),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + 15 = 23 < 25.$$

Otherwise  $F_1^0 \neq \emptyset$  and hence  $|H_1| = |F_1^0| + |F_1^2| \leq 3 + 2 = 5$  and  $|H[W]| \leq 7$ . Again we use the fact that for each  $h \in F_1^0$  the pair  $h \cap W_1$  is nonseparable. Therefore by (3) and (11),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 5 + 7 = 18 < 25.$$

The last case we have to consider is  $H[W] \not\subseteq S_7$ . If  $M \subseteq H[W]$ , then by Lemma 3,  $|H[W]| \leq \text{ex}(7; \{P, C\}|M) = 10$ . Otherwise by Lemma 5,  $|H[W]| \leq \text{ex}^{(2)}(7; \{M, C\}) = 10$ . Hence by (3) and (15),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 13 + 10 = 23 < 25.$$

□

**Fact 6.** For  $n \geq 12$  if  $|W_1| \geq 5$  and  $H_1 \neq \emptyset$  then

$$|H| < \binom{n-6}{2} + 10. \quad (17)$$

*Proof.* The proof is by induction on  $n$  with the initial step  $n = 12$  done in Fact 5. Let  $n \geq 13$ . For  $W_0 = \emptyset$  the inequality (17) results from Fact 2. Otherwise there exist a vertex  $v \in W_0$ . Notice, that since  $|W_1| \geq 5$  we have  $|W_0| \leq n - 10$  and consequently, by (10),  $|H(v)| \leq 4 + \max\{2, |W_0| - 1\} \leq 4 + n - 11 = n - 7$ . Finally, by the induction assumption we get  $|H - v| < \binom{n-7}{2} + 10$ . Therefore,

$$|H| = |H(v)| + |H - v| < n - 7 + \binom{n-7}{2} + 10 = \binom{n-6}{2} + 10.$$

□

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